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QN-spaces, wQN-spaces and covering properties ☆

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Abstract

The main results of the paper are as follows: covering characterizations of wQN-spaces, covering characterizations of QN-spaces and a theorem saying that $C_p(X)$ has the Arkhangel'skiĭ property (α_1) provided that X is a QN-space. The latter statement solves a problem posed by M. Scheepers [M. Scheepers, $C_p(X)$ and Arhangel'skiĭ's α_i-spaces, Topology Appl. 89 (1998) 265–275] and for Tychonoff spaces was independently proved by M. Sakai [M. Sakai, The sequence selection properties of $C_p(X)$, Preprint, April 25, 2006]. As the most interesting result we consider the equivalence that a normal topological space X is a wQN-space if and only if X has the property $S_1(\Gamma_{shr}, \Gamma)$. Moreover we show that X is a QN-space if and only if $C_p(X)$ has the property (α_0), and for perfectly normal spaces, if and only if X has the covering property (β_3).

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1. Introduction

Throughout the paper a topological space means an infinite Hausdorff space.

If $A \subseteq X$, then the *sequential closure* of A is the set $\mathrm{scl}(A)$ of all limits of sequences from the set A. A natural question to ask is whether the sequential closure $\mathrm{scl}(A)$ and the topological closure $\overline{A}$ coincide. If $\mathrm{scl}(A) = \overline{A}$ for any subset $A \subseteq X$, then X is called Fréchet–Urysohn. Otherwise we ask whether

$$\mathrm{scl}\big(\mathrm{scl}(A)\big) = \mathrm{scl}(A). \tag{1}$$

If the equality (1) holds true for any subset of X then X is called an $\mathscr{S}^*$-*space*—see [10].

A.V. Arkhangel'skiĭ [1] introduced four properties (α_i), $i = 1, \ldots, 4$ of a topological space X, which are closely related to the above discussed properties. We are interested in the first and the fourth ones:

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(α_1) For any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x there exists a sequence $\{x_n\}_{n=0}^{\infty}$ converging to x such that the sequence $\{x_n\}_{n=0}^{\infty}$ contains all but finitely many members of the sequence $\{x_{n,m}\}_{m=0}^{\infty}$ for every n.

(α_4) For any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x there exists a sequence $\{x_n\}_{n=0}^{\infty}$ converging to x such that the sequence $\{x_n\}_{n=0}^{\infty}$ contains a member of the sequence $\{x_{n,m}\}_{m=0}^{\infty}$ for infinitely many n.

We introduce another similar property

(α_0) For any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x there exists an unbounded non-decreasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} x_{n_m,m} = x$

and a technical modification:

(α_0^*) For any $x \in X$, for any sequence $\{x_n\}_{n=0}^{\infty}$ converging to x, and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences such that $\lim_{m\to\infty} x_{n,m} = x_n$ for every n, there exists a sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} x_{n_m,m} = x$.

Evidently $(\alpha_1) \to (\alpha_0)$ and $(\alpha_0) \to (\alpha_4)$. For a topological group X the property (α_0) implies also the property (α_0^*). Actually, assuming (α_0) and replacing $x_{n,m}$ by $x_{n,m} - x_n + x$ we obtain a sequence $\{n_m\}_{m=0}^{\infty}$ such that

$$\lim_{m\to\infty}(x_{n_m,m} - x_{n_m} + x) = x.$$

Then also $\lim_{m\to\infty} x_{n_m,m} = x$. Moreover a topological group X possesses the property (α_4) if and only if (1) holds true for any $A \subseteq X$.

By Scheepers [20], a topological space X has the *sequence selection property* if for any sequence of sequences $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of elements of X with $\lim_{m\to\infty} x_{n,m} = x$ for each n, there exists a sequence of natural numbers $\{m_n\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} x_{n,m_n} = x.$$

We remind that the quantifier $(\forall^{\infty} n)$ means "for all but finitely many n", i.e. it is a short notation for $(\exists k)(\forall n \geqslant k)$. Similarly, $(\exists^{\infty} n)$ means "there exist infinitely many n", i.e. $(\forall k)(\exists n > k)$.

If $\mathcal{V}$ is a property of a topological space then instead of saying "X has the property $\mathcal{V}$" we shall often say that "X is a $\mathcal{V}$-space".

Let us recall that a sequence $\{f_n\}_{n=0}^{\infty}$ of real-valued functions is said to *converge quasinormally* to a function f on X, written $f_n \xrightarrow{\mathrm{QN}} f$ on X (see [9,2]), if there exists a *control sequence* $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive reals converging to 0 such that

$$(\forall x \in X)(\forall^{\infty} n) \quad \bigl|f_n(x) - f(x)\bigr| < \varepsilon_n.$$

"$f_n \to f$ on X" ("$f_n \rightrightarrows f$ on X") means that f_n converges to f on X pointwise (uniformly). For a topological space X, we denote by $C_p(X)$ the topological space of all real-valued continuous functions defined on X with the topology of pointwise convergence, i.e. the relativised product topology of the product space ${}^X\mathbb{R}$. A sequence $\{f_n\}_{n=0}^{\infty}$ of elements of $C_p(X)$ converges to f in this topology if and only if it converges pointwise on X.

In [7] inspired by some problems from the theory of trigonometric thin sets (compare e.g. [6]), the authors introduced the notions of a QN-space and a wQN-space. A topological space X is said to be a QN-*space* if every sequence $f_n \in C_p(X), n \in \omega$ converging to 0 on X pointwise converges also quasinormally. A topological space X is a wQN-*space* if from every sequence $f_n \in C_p(X), n \in \omega$ converging to 0 on X pointwise one can choose a subsequence converging to 0 quasinormally. Basic properties of a QN-space and a wQN-space were investigated in [7,8].

M. Scheepers [21] has shown that in $C_p(X)$, the properties (α_2), (α_3), (α_4) are equivalent. By results of Scheepers [22] and Fremlin [11,12] we have (compare [3])

Theorem 1 *(D. Fremlin – M. Scheepers). For a topological space X, the following are equivalent*:

(a) *$C_p(X)$ has the property (α_4).*
(b) *$C_p(X)$ has the sequence selection property.*
(c) *X is a* wQN*-space.*
(d) *$C_p(X)$ is an $\mathcal{S}^*$-space.*

Let us note that in Fremlin [11] a space X is called an s_1-space if $C_p(X)$ is an $\mathcal{S}^*$-space.
Scheepers [21] also showed

Theorem 2. *If $C_p(X)$ has the property (α_1) then X is a* QN*-space.*

Scheepers [21] immediately raised the following problem.

Problem 3. Is it true that if a set X of real numbers is a QN-space, then the function space $C_p(X)$ has the property (α_1)?

In Theorem 11 of this paper we answer the problem affirmatively.

After the first draft of the paper was sent to some fellows and the main results were included in the survey article [4], we obtained the preprint [18] by Masami Sakai in which, among another results, he proves a slightly weaker variant of the equivalence $(a) \equiv (d)$ of our Theorem 11, and a slightly stronger variant of the equivalence $(c) \equiv (f)$ of our Theorem 17. Anyway the proofs are essentially different: Our main aim was to solve Scheepers' conjecture 8, whereas M. Sakai has concentrated on the weak sequence selection property introduced in [20].

2. Covering properties

A family $\mathcal{U}$ of subsets of a set X is called a *cover* of X if $X \notin \mathcal{U}$ and $\bigcup \mathcal{U} = X$. Note that we allow that $\emptyset$ is a member of a cover. If every member of a cover $\mathcal{U}$ is an open set then $\mathcal{U}$ is an *open cover*. Similarly for closed and clopen covers, respectively. A cover $\mathcal{V}$ is a *refinement* of a cover $\mathcal{U}$ if

$$(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}) \quad V \subseteq U.$$

A cover $\mathcal{V}$ is a *regular refinement* if

$$(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}) \quad \overline{V} \subseteq U.$$

A sequence $\mathcal{U}_n$, $n \in \omega$ of covers of a topological space X is said to be a *refining sequence of covers* if every $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$. Similarly, a sequence $\mathcal{U}_n$, $n \in \omega$ of covers of a topological space X is said to be a *regular refining sequence of covers* if every $\mathcal{U}_{n+1}$ is a regular refinement of $\mathcal{U}_n$.

An infinite cover $\mathcal{U}$ of X is called a *γ-cover* if every $x \in X$ is in all but finitely many members of $\mathcal{U}$. If a γ-cover $\mathcal{V}$ is a refinement of a cover $\mathcal{U}$, $U \in \mathcal{U}$, then $V \subseteq U$ holds true only for finitely many $V \in \mathcal{V}$. A γ-cover $\mathcal{U}$ is said to be *shrinkable* if there exists a closed γ-cover that is a refinement of $\mathcal{U}$. Note that every infinite subset of a γ-cover is a γ-cover. That is not true for shrinkable covers, however every shrinkable cover contains a countable shrinkable subcover. A cover $\mathcal{U}$ of X is called an *ω-cover* if every finite $A \subseteq X$ is contained in some $U \in \mathcal{U}$. Note that also an ω-cover is infinite since we assume that X is infinite. A topological space is called a *γ-space* if every ω cover contains a γ-subcover. We denote by $\Gamma(X)$, or simply Γ the family of all open γ-covers of X. Γ_{shr} will denote the family of all open shrinkable γ-covers and $\Omega(X)$ will denote the family of all open ω-covers of X.

Note that if $\mathcal{U}_n$, $n \in \omega$ is a regular refining sequence of γ-covers then every cover $\mathcal{U}_n$ is shrinkable.

One can easily see that any two countable γ-covers have a common refinement that is again a γ-cover. Also any two countable shrinkable γ-covers have a common refinement that is a shrinkable cover. Actually, if $\{U_n\colon n \in \omega\}$, $\{V_n\colon n \in \omega\}$ are γ-covers then $\{U_n \cap V_n\colon n \in \omega\}$ is a γ-cover which is a common refinement of both $\{U_n\colon n \in \omega\}$ and $\{V_n\colon n \in \omega\}$. The same holds true for open, closed, clopen covers, respectively.

M. Scheepers [19] introduced some covering properties of a topological space. Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of X. The *property* $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ says the following: for every sequence $\mathcal{U}_n$, $n \in \omega$ of elements of $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n\colon n \in \omega\} \in \mathcal{B}$. One can easily see that X has the property $\mathrm{S}_1(\Gamma, \Gamma)$ if for every refining sequence of open γ-covers $\mathcal{U}_n$, $n \in \omega$ there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n\colon n \in \omega\}$ is a γ-cover. Similarly for the property $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$.

Analogously to such formulation we introduce a weaker property. We shall say that X has the *property* $\overline{\mathrm{S}}_1(\Gamma,\Gamma)$ if for every regular refining sequence of γ-covers $\mathcal{U}_n$, $n\in\omega$ there exist sets $U_n\in\mathcal{U}_n$ such that $\{U_n\colon n\in\omega\}$ is a γ-cover. The question whether $C_p(X)$ is a Fréchet–Urysohn space was answered for Tychonoff spaces by Gerlits and Nagy [13] showing that $C_p(X)$ is a Fréchet–Urysohn space if and only if X is a γ-space. The last property can be considered as a covering property of the topological space X. Moreover Gerlits and Nagy [13] showed

Theorem 4. *If X is a Tychonoff space then $C_p(X)$ is a Fréchet–Urysohn if and only if X has the property* $\mathrm{S}_1(\Omega,\Gamma)$.

There is a natural question whether there exists a covering property of X that is equivalent to the property of $C_p(X)$ to be an $\mathscr{S}^*$-space or to possess the property (α_1), respectively.

By way let us note the following. Hurewicz [16] introduced the property H^{**} of a topological space X: for any sequence $\{f_n\}_{n=0}^{\infty}$ of continuous functions from X into $\mathbb{R}$ the family of sequences of reals $\{\{f_n(x)\}_{n=0}^{\infty}\colon x\in X\}$ is eventually bounded, i.e. there exists a sequence of reals $\{r_n\}_{n=0}^{\infty}$ such that

$$(\forall x\in X)(\forall^{\infty}n)\quad f_n(x)<r_n.$$

Hurewicz [16] essentially proved a covering characterization of this property (for details see [3]):

Theorem 5. *A perfectly normal space X possesses the property* H^{**} *if and only if X possesses the following covering property* E^{**}_{ω}: *for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of countable open covers of X there exist finite subsets $\mathcal{V}_n\subseteq\mathcal{U}_n$ such that*

$$(\forall x)(\forall^{\infty}n)\quad x\in\bigcup\mathcal{V}_n.$$

Moreover, in [8] the authors introduced a notion of an mQN-*space*: X is an mQN-space if for any sequence $\{f_n\}_{n=0}^{\infty}$ of continuous functions $f_n\to 0$ on X and such that $f_{n+1}(x)\leqslant f_n(x)$ for any $x\in X$ and any $n\in\omega$ one has $f_n\xrightarrow{\mathrm{QN}}0$ on X. For metric separable spaces in [8] and for any perfectly normal spaces in [5] the authors found a covering characterization of an mQN-space.

Theorem 6. *A perfectly normal space X is an* mQN-*space if and only if X possesses the property* E^{**}_{ω}.

In this paper we present covering characterizations of a wQN-space and a QN-space, i.e. covering characterizations of a topological space X for which $C_p(X)$ is an $\mathscr{S}^*$-space or possesses the property (α_1), respectively. We suppose that in some sense our results are similar to Theorem 6 and to those by Gerlits and Nagy [13].

3. Scheepers' conjecture

Scheepers [22] proved

Theorem 7 *(M. Scheepers). Any* $\mathrm{S}_1(\Gamma,\Gamma)$*-space is a wQN-space.*

Then he conjectured that

Conjecture 8 *(M. Scheepers). A* wQN-*space possesses property* $\mathrm{S}_1(\Gamma,\Gamma)$.

In [14] the author proves a modification of this conjecture. Let Γ_{clopen} denote the family of all clopen γ-covers. By [14], a topological space X is an nCM-*space* if X cannot be continuously mapped onto the unit interval $\mathbb{I}$. By [7] a normal nCM-space possesses the following property:

$$(\forall A\text{ closed})(\forall U\supseteq A\text{ open})(\exists W\text{ clopen})\quad (A\subseteq W\subseteq U). \tag{2}$$

M. Sakai [18] used this property as a condition about the large inductive dimension $\mathrm{Ind}(X)=0$. Then, see [1] [15] and Theorem 43 in [4], we obtain

[1] The result immediately follows from the proofs of Lemmas 1 and 2 of [14] and is explicitly stated in J. Haleš' thesis.

Theorem 9. *For a normal topological space X, the following are equivalent*:

(a) *X is a* wQN-*space.*
(b) *X is an* $\mathrm{S}_1(\Gamma_{\mathrm{clopen}}, \Gamma)$*-space and X cannot be continuously mapped onto the unit interval* $\mathbb{I}$.
(c) *X is a* $\mathrm{S}_1(\Gamma_{\mathrm{clopen}}, \Gamma)$*-space satisfying the condition* (2).

For Tychonoff spaces M. Sakai [18] proves that the statement (a) is equivalent to a condition which, for a normal space, is equivalent to the statement (c).

Theorem 9 can be considered as a covering characterization of wQN-spaces. In spite of the fact that the statement (b) could be hardly considered as a covering property one cannot doubt that the statement (c) is a covering property. Anyway we present another covering characterization of wQN-spaces. Actually, modifying a proof of Theorem 9 we show that the property $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$ characterizes wQN-spaces. Thus Scheepers' conjecture is equivalent to the implication $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma) \to \mathrm{S}_1(\Gamma, \Gamma)$. Since we suppose that the property $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$ is not "seemingly" but "actually" weaker than $\mathrm{S}_1(\Gamma, \Gamma)$, we begin to doubt that Scheepers' conjecture 8 is true.

Let us introduce some technical details. A sequence $\{f_n\}_{n=0}^{\infty}$ *discretely converges* to f on X, written $f_n \xrightarrow{\mathrm{D}} f$ on X (compare [9]), if

$$(\forall x \in X)(\forall^{\infty} n) \quad f_n(x) = f(x).$$

Evidently if $f_n \xrightarrow{\mathrm{D}} f$ then also $f_n \xrightarrow{\mathrm{QN}} f$.

A space $C_p(X)$ has the *discrete sequence selection property* if for every sequence $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ such that $f_{n,m} \xrightarrow{\mathrm{D}} 0$, $m \to \infty$ on X for every $n \in \omega$, there exists a sequence of natural numbers $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X.

We are ready to prove one of the main results of the paper.

Theorem 10. *For a normal topological space X, the following are equivalent*:

(a) *X is a* wQN-*space.*
(b) *$C_p(X)$ has the sequence selection property.*
(c) *$C_p(X)$ has the discrete sequence selection property.*
(d) *X is an* $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$*-space.*
(e) *X is an* $\bar{\mathrm{S}}_1(\Gamma, \Gamma)$*-space.*

Proof. The implication (a) $\to$ (b) was proved by Fremlin [12]. Evidently (b) implies (c) and (d) implies (e).

We assume that X has the discrete sequence selection property and we want to show that X is an $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$-space. Let $\mathcal{U}_n$, $n \in \omega$ be a sequence of open shrinkable γ-covers. Then for every n there exists a closed γ-cover $\mathcal{V}_n$ that is a refinement on the cover $\mathcal{U}_n$. Take $\{Z_{n,m}\colon m \in \omega\} \subseteq \mathcal{V}_n$ with a bijective enumeration. For every n, m let $U_{n,m} \in \mathcal{U}_n$ be such that $Z_{n,m} \subseteq U_{n,m}$. Since X is normal there exists a continuous function $f_{n,m} : X \to \mathbb{I}$ such that $f_{n,m}(x) = 1$ for $x \in X \setminus U_{n,m}$ and $f_{n,m}(x) = 0$ for $x \in Z_{n,m}$.

Assume $x \in X$, $n \in \omega$. Since $\{Z_{n,m}\colon m \in \omega\}$ is a γ-cover there exists an m_0 such that $x \in Z_{n,m}$ for every $m \geqslant m_0$. Then also $f_{n,m}(x) = 0$ for every $m \geqslant m_0$. Thus $f_{n,m} \xrightarrow{\mathrm{D}} 0$ on X for every n. By the discrete sequence selection property there exists a sequence $\{m_n\}_{n=0}^{\infty}$ such that $f_{n,m_n} \to 0$ on X. If $x \in X$ then there exists an n_0 such that $f_{n,m_n} < 1$ for any $n \geqslant n_0$. Thus $x \in U_{n,m_n}$ for $n \geqslant n_0$. If $\{U_{n,m_n}\colon n \in \omega\}$ were finite then $U_{n,m_n} = U \neq X$ for infinitely many n's, what is impossible. Thus the set $\{U_{n,m_n}\colon n \in \omega\}$ is the γ-cover of X as desired.

Now we assume that (e) holds true and we show that X is a wQN-space. So let $f_m \to 0$ on X. We can assume that $f_m(x) > 0$ for every m and every $x \in X$. Let $\varepsilon_n \to 0$ be a strictly decreasing sequence of positive reals. We denote

$$U_{n,m} = \{x \in X\colon\ f_m(x) < \varepsilon_n\} \qquad \mathcal{U}_n = \{U_{n,m}\colon m \in \omega\}, \qquad A = \{n \in \omega\colon\ X \notin \mathcal{U}_n\}.$$

If $n \in A$ then $\mathcal{U}_n$ is a γ-cover. Actually, since $f_m \to 0$ on X, for every finite subset $F \subseteq X$ there exists an $m_0 \in \omega$ such that $F \subseteq U_{n,m}$ for all $m \geqslant m_0$. Since $X \notin \mathcal{U}_n$, this implies that $\mathcal{U}_n$ is infinite. Hence $\mathcal{U}_n$ is a γ-cover.

If $\omega \setminus A = \{n_k\colon k \in \omega\}$ is an infinite set then there exists a sequence $\{m_k\}_{k=0}^{\infty}$ such that $U_{n_k,m_k} = X$. If the sequence $\{m_k\}_{k=0}^{\infty}$ were bounded then $f_m = 0$ for some m. Thus we can assume that both sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ are strictly increasing. Since $f_{m_k}(x) < \varepsilon_{n_k}$ for every $x \in X$ we can conclude that $f_{m_k} \rightrightarrows 0$ on X.

If $\omega \setminus A$ is finite then we can omit corresponding covers and assume that $A = \omega$. If $n < k$ then evidently $\overline{U}_{k,m} \subseteq U_{n,m}$ for every m. Thus $\mathcal{U}_{n+1}$ is a regular refinement of $\mathcal{U}_n$ for every n. Then there exists a sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{U_{n,m_n}\colon n \in \omega\}$ is a γ-cover. If $\{m_n\}_{n=0}^{\infty}$ were bounded then $m_n = m$ for infinitely many n and one can easily find an $x \in X$ such that x does not belong to infinitely many U_{n,m_n}, what is impossible. Therefore we can assume that the sequence $\{m_n\}_{n=0}^{\infty}$ is strictly increasing. Then $f_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X with the control sequence $\{\varepsilon_n\}_{n=0}^{\infty}$. □

4. QN-space and (α_1)

Using the idea by D. Fremlin [12] we give an affirmative answer to Scheepers' Problem 3. Moreover we show that for $C_p(X)$ the property (α_1) is equivalent to its modifications introduced in Section 1.

Theorem 11. *For a topological space X, the following are equivalent*:

(a) *$C_p(X)$ has the property (α_1)*;
(b) *$C_p(X)$ has the property (α_0)*;
(c) *$C_p(X)$ has the property (α_0^*)*;
(d) *X is a* QN-*space.*

Proof. The implications (a) → (b) and (b) → (c) are trivial.

We show that (c) implies (d). Assume that $f_n \to 0$ pointwise on X. We denote $f_{n,m} = 2^n \cdot |f_m| + 2^{-n}$. Then $f_{n,m} \to 2^{-n}$ for $m \to \infty$ and for every n. By (c) there exists a sequence of natural numbers $\{k_m\}_{m=0}^{\infty}$ such that $f_{k_m,m} \to 0$ on X. Then

$$(\forall x \in X)(\forall^{\infty} m) \quad f_{k_m,m}(x) < 1.$$

Thus for all but finitely many m we have $|f_m(x)| < 2^{-k_m}$. Since $2^{-k_m} \to 0$ we obtain $f_m \xrightarrow{\mathrm{QN}} 0$ on X.

Finally we show that (d) implies (a). Assume that X is a QN-space. We want to show that $C_p(X)$ has the property (α_1). So let $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ be a sequence of sequences converging to 0 on X. We can assume that values of each $f_{n,m}$ are in $\mathbb{I}$. We define

$$h_{n,m}(x) = \min\{2^{-n-1}, f_{n,m}(x)\},$$
$$g_m(x) = \sum_{n=0}^{\infty} h_{n,m}(x),$$

for every $x \in X$. Evidently g_m are continuous and $g_m \to 0$ on X. Actually, if $x \in X$ and $\varepsilon > 0$ are given then there exists an n_0 such that $\sum_{n=n_0}^{\infty} 2^{-n-1} < \varepsilon/2$. Take m_0 such that $f_{n,m}(x) < \varepsilon/(2n_0)$ for $m \geqslant m_0$ and $n < n_0$. Then $g_m(x) < \varepsilon$ for $m \geqslant m_0$.

Since X is a QN-space there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive reals, $\varepsilon_n \to 0$ such that

$$(\forall x)(\exists l_x)(\forall m \geqslant l_x) \quad g_m(x) < \varepsilon_m.$$

There are also natural numbers m_k such that

$$(\forall k)(\forall m \geqslant m_k) \quad \varepsilon_m < 2^{-k-1}.$$

We can assume that $m_k \leqslant m_{k+1}$ for any k. We claim that the sequence (in any order)

$$\{f_{n,m}\colon n \in \omega \wedge m \geqslant m_n\} \tag{3}$$

converges to 0 on X.

Let $x \in X$ and $\varepsilon > 0$. Take such an k_0 that $2^{-k_0-1} < \varepsilon$ and $m_{k_0} > l_x$. Moreover, take such a natural number p that $f_{n,m}(x) < \varepsilon$ for $m \geqslant p$ and any $n < k_0$. If $n \geqslant k_0$ and $m \geqslant m_n \geqslant m_{k_0} > l_x$ then

$$g_m(x) < \varepsilon_m < 2^{-n-1}$$

and therefore

$$h_{n,m}(x) < 2^{-n-1} \leqslant 2^{-k_0-1} < \varepsilon.$$

Thus $f_{n,m}(x) < \varepsilon$ for any $n \geqslant k_0$, any $m \geqslant m_n$ and for any $n < k_0$ and any $m \geqslant p$. Thus for all members $f_{n,m}$ of the sequence (3) but those with $n < k_0$ and $m < p$ we have $f_{n,m}(x) < \varepsilon$. □

For rather technical reasons we introduce a property of a topological space X:

(A_1^d) for any sequence $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging discretely to 0 on X there exists a sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that

$$(\forall x \in X)(\forall \varepsilon > 0)\big(\forall^{\infty}[n,m]\big) \quad \big(m > m_n \to \big|f_{n,m}(x)\big| < \varepsilon\big).$$

Evidently if $C_p(X)$ has the property (α_1) then X has the property (A_1^d).

5. Covering characterizations of a QN-space

Assume that $U_n, n \in \omega$ is an enumeration of a countable cover $\mathcal{U}$ of X, i.e. $\mathcal{U} = \{U_n\colon n \in \omega\}$. The enumeration $U_n, n \in \omega$ is called *adequate* if every element $U \in \mathcal{U}$ occurs only finitely many times in the enumeration. If

$$(\forall x \in X)(\forall^{\infty} n) \quad x \in U_n, \tag{4}$$

then $\mathcal{U}$ is a γ-cover and the enumeration is adequate. Vice versa, every adequate enumeration of a countable γ-cover satisfies the condition (4). Consequently if $U_n \subseteq V_n$ for every $n \in \omega$ and $U_n, n \in \omega$ is an adequate enumeration of a γ-cover then also $V_n, n \in \omega$ is an adequate enumeration of a γ-cover. Moreover, if $U_n, n \in \omega$ and $V_n, n \in \omega$ are adequate enumerations of γ-covers then $U_n \cap V_n, n \in \omega$ is an adequate enumeration of a γ-cover.

Let $\mathcal{U}_n$, $n \in \omega$ be a sequence of countable covers of X. An enumeration $U_{n,m}, n, m \in \omega$ of the sequence $\mathcal{U}_n$, $n \in \omega$ is *adequate* if $U_{n,m}, m \in \omega$ is adequate for every n. If every $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ then an adequate enumeration $U_{n,m}, n, m \in \omega$ of the sequence $\mathcal{U}_n$, $n \in \omega$ is *coherent* if $U_{n+1,m} \subseteq U_{n,m}$ for any $n, m \in \omega$. If $\mathcal{U}_n$, $n \in \omega$ is a regular refining sequence of countable covers then a coherent enumeration $U_{n,m}, n, m \in \omega$ is called *regularly coherent* if $\overline{U}_{n+1,m} \subseteq U_{n,m}$ for any $n, m \in \omega$.

We introduce three covering properties of a topological space:

(β_1) For every sequence of countable open γ-covers $\mathcal{U}_n$, $n \in \omega$ there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n (\mathcal{U}_n \setminus \mathcal{V}_n)$ is a γ-cover of X.

(β_2) For every sequence of countable open γ-covers $\mathcal{U}_n$, $n \in \omega$ with an adequate enumeration $U_{n,m}, n, m \in \omega$ there exists a non-decreasing unbounded sequence $\{n_m\}_{m=0}^{\infty}$ of natural numbers such that $U_{n_m,m}, m \in \omega$ is an adequate enumeration of a γ-cover of X.

(β_3) For every sequence of countable open γ-covers $\mathcal{U}_n$, $n \in \omega$ with an adequate enumeration $U_{n,m}, n, m \in \omega$ there exists an open γ-cover $\{V_m\colon m \in \omega\}$ such that

$$(\forall n)(\forall^{\infty} m) \quad V_m \subseteq U_{n,m}.$$

Moreover, we introduce two versions which depend on the enumeration of considered coverings.

(β_1^*) For every regular refining sequence of countable open γ-covers $\mathcal{U}_n$, $n \in \omega$ with a regularly coherent enumeration $U_{n,m}, n, m \in \omega$ there exists a sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that $\{U_{n,m}\colon n \in \omega \wedge m \geqslant m_n\}$ is a γ-cover of X.

(β_2^*) For every regular refining sequence of countable open γ-covers $\mathcal{U}_n$, $n \in \omega$ with a regularly coherent enumeration $U_{n,m}, n, m \in \omega$ there exists an unbounded non-decreasing sequence $\{n_m\}_{m=0}^{\infty}$ of natural numbers such that $U_{n_m,m}, m \in \omega$ is an adequate enumeration of a γ-cover of X.

We begin with auxiliary results.

Lemma 12. *Every normal topological space with the property* (A_1^d) *has the property* (β_1^*).

Proof. We assume that a normal topological space X has the property (A_1^d) and that $\mathcal{U}_n$, $n \in \omega$ is a regular refining sequence of countable open γ-covers with a regularly coherent enumeration $U_{n,m}, n, m \in \omega$. Using normality of X there exist continuous functions $f_{n,m}: X \to \mathbb{I}$ such that $f_{n,m}(x) = 0$ for $x \in \overline{U}_{n+1,m}$ and $f_{n,m}(x) = 1$ for $x \in X \setminus U_{n,m}$. As in the proof of theorem 10 one can easily see that $f_{n,m} \xrightarrow{\mathrm{D}} 0$ on X for every n. By (A_1^d) there exists a sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that the sequence $f_{n,m}, n \in \omega, m \geqslant m_n$ converges to 0 on X. We show that $\{U_{n,m}\colon n \in \omega \wedge m \geqslant m_n\}$ is a γ-cover.

Actually, if $x \in X$ then $f_{n,m}(x) < 1$ for all but finitely many couples $[n,m]$ such that $n \in \omega$ and $m \geqslant m_n$. Therefore also $x \in U_{n,m}$ for all but finitely many couples $[n,m]$ such that $n \in \omega$ and $m \geqslant m_n$. □

Lemma 13. *Every* (β_1^*)*-space has the property* (β_2^*).

Proof. Assume X has the property (β_1^*). Let $\mathcal{U}_n$, $n \in \omega$ be a regular refining sequence of countable open γ-covers with a regularly coherent enumeration $U_{n,m}, n, m \in \omega$. By (β_1^*) there exists a sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that $\mathcal{U} = \{U_{n,m}\colon n \in \omega \wedge m \geqslant m_n\}$ is a γ-cover of X. We can assume that the sequence $\{m_n\}_{n=0}^{\infty}$ is increasing.

We construct by induction an increasing sequence $\{l_i\}_{i=0}^{\infty}$ and an unbounded non-decreasing sequence $\{n_m\}_{m=0}^{\infty}$ as follows. Set $n_m = 0$ for $m < m_0$ and $l_0 = m_0$. Find an $l_{k+1} \geqslant m_{k+1}$ such that for no $i \geqslant l_{k+1}$ the set $U_{k+1,i}$ is among the sets $U_{n_0,0}, \ldots, U_{n_k,k}$ and set $n_m = k+1$ for $l_k \leqslant m < l_{k+1}$.

One can easily see that the sequence $U_{n_k,k}, k \in \omega$ is an adequate enumeration of a γ-cover. □

Lemma 14. *Every topological space with property* (β_2^*) *is a* QN*-space.*

Proof. Assume that X has the property (β_2^*). Let $f_m \to 0$ on X. We can assume that $0 < f_m(x) \leqslant 1$ for every $x \in X$ and every m. Let $x_n, n \in \omega$ be mutually different elements of X. For any $n, m \in \omega$ we define

$$U_{n,m} = \{x \in X\colon f_m(x) < 2^{-n} \wedge x \neq x_m\}, \qquad \mathcal{U}_n = \{U_{n,k}\colon k \in \omega\}.$$

It is easy to see that $\mathcal{U}_n$, $n \in \omega$ is an open γ-cover with a regularly coherent enumeration $U_{n,m}, n, m \in \omega$. By (β_2^*) there exists an unbounded non-decreasing sequence $\{n_m\}_{m=0}^{\infty}$ of natural numbers such that $U_{n_m,m}, m \in \omega$ is an adequate enumeration of a γ-cover. Thus $x \in X$ belongs to all but finitely many members of $\{U_{n_m,m}\colon m \in \omega\}$, i.e. there exists an m_0 such that $x \in U_{n_m,m}$ for any $m \geqslant m_0$. Then

$$(\forall x \in X)(\forall^\infty m) \quad f_m(x) < 2^{-n_m}.$$

Since $n_m \to \infty$ we obtain $f_m \xrightarrow{\mathrm{QN}} 0$ on X. □

The properties (β_1^*) and (β_2^*) are rather technical. The properties (β_1), (β_2), and (β_3) are more invariant. We show that all they are equivalent.

Let us recall that a topological space X is called a *σ-space* if every F_σ-subset of X is also a G_δ-set. By Recław [17] every perfectly normal QN-space is a σ-space. I. Recław proved it for metric spaces only. However one can easily see, compare [14,18], that a similar argument works for a perfectly normal space too.

By [7] a QN-space satisfies the condition (2), see also Theorem 9. Using this we obtain a slight modification of Lemma 5 of [14].

Lemma 15. *If X is a perfectly normal* QN*-space then for every open countable γ-cover of X with an adequate enumeration $U_n, n \in \omega$ of X there exist clopen sets $V_n \subseteq U_n$, $n \in \omega$ such that $V_n, n \in \omega$ is an adequate enumeration of a γ-cover of X.*

Proof. The sequence $G_n = \bigcap_{k \geqslant n} U_k, n \in \omega$ is a non-decreasing γ-cover by G_δ-sets. There exist closed sets $F_{n,m}$ such that $G_n = \bigcup_m F_{n,m}$. We can assume $F_{n,m} \subseteq F_{n,m+1}$ and $F_{n,m} \subseteq F_{n+1,m}$ for every $n, m \in \omega$. Then $F_{n,n}, n \in \omega$ is a non-decreasing closed γ-cover. By (2) there exist clopen sets V_n such that $F_{n,n} \subseteq V_n \subseteq U_n$. It is easy to see that $V_n, n \in \omega$ is an adequate enumeration of a γ-cover of X. □

We use the lemma to prove

Lemma 16. *Every perfectly normal space with the property* (β_1^*) *has the property* (β_1).

Proof. Assume that a perfectly normal space X has the property (β_1^*). Let $\mathcal{U}_n, n \in \omega$ be a sequence of countable open γ-covers. For every n let $U_{n,m}, m \in \omega$ be a bijective enumeration of $\mathcal{U}_n$. By Lemmas 13 and 14 a space with the property (β_1^*) is a QN-space. Thus by Lemma 15 there are clopen sets $V_{n,m} \subseteq U_{n,m}$ such that $V_{n,m}, m \in \omega$ is an adequate enumeration of a γ-cover for every n. Setting

$$W_{n,m} = \bigcap_{i=0}^{n} V_{i,m} \tag{5}$$

we obtain that $W_{n,m} \subseteq U_{n,m}$ and $W_{n,m}, n, m \in \omega$ is a regularly coherent enumeration of a regular refining sequence of γ-covers. By (β_1^*) there is a sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{W_{n,m}\colon m \geqslant m_n \wedge n \in \omega\}$ is a γ-cover. Then also $\{U_{n,m}\colon m \geqslant m_n \wedge n \in \omega\}$ is a γ-cover. $\square$

Theorem 17. *Let X be a perfectly normal topological space. Then the following conditions are equivalent*:

(a) $C_p(X)$ *has the property* (α_1).
(b) $C_p(X)$ *has the property* (A_1^d).
(c) X *has the property* (β_1).
(d) X *has the property* (β_2).
(e) X *has the property* (β_3).
(f) X *is a* QN-*space.*

Proof. The implications (a) → (b) and (e) → (c) are trivial. The implication (b) → (c) follows by Lemmas 12 and 16. By Theorem 11 we have the implication (f) → (a). Evidently $(\beta_2) \to (\beta_2^*)$. Thus by Lemma 14 we obtain (d) → (f).

We show that (c) → (d). So assume (β_1). Evidently $(\beta_1) \to (\beta_1^*)$. Thus by Lemma 13 we have (β_2^*) and therefore by Lemma 14, X is a QN-space. Now let $U_{n,m}, n, m \in \omega$ be an adequate enumeration of a sequence of countable open γ-covers of X. Similarly as in the proof of Lemma 16 we construct clopen sets $W_{n,m} \subseteq U_{n,m}$ such that $W_{n,m}\colon n, m \in \omega$ is a regularly coherent enumeration of a regular refining sequence of γ-covers. By (β_2^*) there exists an unbounded non-decreasing sequence $\{n_m\}_{n=0}^{\infty}$ such that $W_{n_m,m}\colon m \in \omega$ is an adequate enumeration of a γ-cover of X. Then $U_{n_m,m}\colon m \in \omega$ is also an adequate enumeration of a γ-cover of X.

We show the implication (d) → (e). Let $\mathcal{U}_n, n \in \omega$ be a sequence of countable open γ-covers with an adequate enumeration $U_{n,m}, n, m \in \omega$. We denote

$$W_{n,m} = \bigcap_{i \leqslant n} U_{i,m}, \qquad \mathcal{W}_n = \{W_{n,m}\colon m \in \omega\}.$$

Then $\mathcal{W}_n$ is a γ-cover and a refinement of $\mathcal{U}_n$. By induction we obtain that $W_{n,m}, n, m \in \omega$ is a coherent enumeration. Then by (d) there is a non-decreasing unbounded sequence $\{n_m\}_{m=0}^{\infty}$ such that $W_{n_m,m}, m \in \omega$ is an adequate enumeration of a γ-cover. Since $W_{n_m,m} \subseteq U_{n,m}$ for every m such that $n_m \geqslant n$ we are done. $\square$

Corollary 18. *For a perfectly normal space all properties* (β_1), (β_2), (β_3), (β_1^*), *and* (β_2^*) *are equivalent.*

We have already remarked that a perfectly normal QN-space X is a σ-space. Thus by [7] every subset of X is again a QN-space. By Theorem 17 also every subset of X has any of considered properties.

6. Problems

We still consider as the main problem Scheepers' Conjecture 8. As we have already remarked we can formulate it equivalently as

Problem 19. For a perfectly normal topological space X, does the property $\mathrm{S}_1(\Gamma, \Gamma)$ follow from the property $\mathrm{S}_1(\Gamma_{\mathrm{shr}}, \Gamma)$?

There is another equivalent formulation of this problem.

Problem 20. Does every nCM-space with the property $\mathrm{S}_1(\Gamma_{\mathrm{clopen}}, \Gamma)$ have the property $\mathrm{S}_1(\Gamma, \Gamma)$?

We present a fine question. In some steps of our proof of Theorem 17 we needed the assumption that the topological space X is perfectly normal.

Problem 21. Which of assertions of Theorem 17 are equivalent for any normal or even any Hausdorff space?

With a kind permission of Boaz Tsaban we present an interesting question raised by him. Let Γ_{Borel} denote the family of all countable Borel covers of X. In [23] Theorem 1, the authors proved that X is a $\mathrm{S}_1(\Gamma_{\mathrm{Borel}}, \Gamma_{\mathrm{Borel}})$-space if and only if every Borel measurable image of X into the Baire space ${}^{\omega}\omega$ is eventually bounded. Assume now that X is a $\mathrm{S}_1(\Gamma_{\mathrm{Borel}}, \Gamma_{\mathrm{Borel}})$-space and $U_{n,m}, n, m \in \omega$ is an adequate enumeration of a sequence of open γ-covers. The mapping ψ defined by

$$\psi(x)(n) = \min\{m\colon\ (\forall k \geqslant m)\ x \in U_{n,k}\}$$

is Borel measurable. If $\{m_n\}_{n=0}^{\infty}$ is the upper bound of the set $\psi(X) \subseteq {}^{\omega}\omega$ then we obtain the property (β_2) and therefore X is a QN-space.

Problem 22. Can a QN-space have an unbounded Borel image in the Baire space ${}^{\omega}\omega$?

We close with an old basic problem.

Problem 23. Can we find in **ZFC** a wQN-space that is not a QN-space?

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